International Journal of Theoretical Physics, Vol. 24, No. 7, 1985

Geometrical Interpretation of Inönü–Wigner Contractions

Eric A. Lord[1]

Received February 16, 1985

The Inönü–Wigner contractions which interrelate the Lie algebras of the isometry groups of metric spaces are discussed with reference to deformations of the absolutes of the spaces. A general formula is derived for the Lie algebra commutation relations of the isometry group for any N-dimensional metric space. These ideas are illustrated by a discussion of important particular cases, which interrelate the four-dimensional de Sitter, Poincaré, and Galilean groups.

1. INTRODUCTION

Recently Sanjuan (1984) has investigated the geometrical significance of the Inönü–Wigner contractions that interrelate the Lie algebras of the isometry groups of the nine Cayley–Klein planes. We shall relate this idea to the projective characterization of metric planes, whereby a metric plane is obtained from a projective plane by specializing a conic and its envelope. The generalization to higher dimensions will then be described, and finally illustrated by a discussion of some particularly interesting special cases.

2. PROJECTIVE CHARACTERIZATION OF THE CAYLEY–KLEIN PLANES

Let Q be a nonsingular conic in a real projective plane $P(2)$. Call the points on Q *isotropic* points (or "points at infinity") and the tangents to Q isotropic lines (or "null lines"). Define the *distance* between two nonisotropic points to be $\kappa \ln \chi$, where χ is the cross ratio determined by the two points and the two isotropic points collinear with them, and κ is an arbitrary constant. Define the *angle* between two nonisotropic lines to be $\kappa' \ln \chi$, where χ is the cross ratio determined by the two lines and the two

[1]Department of Applied Mathematics, Indian Institute of Science, Bangalore 560 012, India.

0020-7748/85/0700-0723$04.50/0

isotropic lines concurrent with them, and κ' is a constant. The distance measure on a line is *elliptic* or *hyperbolic* according as the isotropic points on it are conjugate complex or real and distinct (two different constants κ can be used for these two cases). If the line is isotropic the two isotropic points on it coincide and hence all distances on an isotropic line are zero. The measure of angle about a point is elliptic or hyperbolic according as the isotropic lines through it are conjugate complex or real and distinct (two different constants κ'). If the point is isotropic the two isotropic lines though it coincide and the angle between any two lines whose intersection is isotropic ("at infinity") is zero. Lines intersecting at infinity are called parallel. The trivial distance measure on an isotropic lines and the trivial angle measure about an isotropic point will be called *null* measures.

The set of all nonisotropic points and lines now constitute a metric plane. The quadric Q is its *absolute.*

The foregoing is just a brief sketch of the standard projective approach to metrical concepts. The reader is referred to projective geometry textbooks for further details.

Let the homogeneous coordinates of a point x in $P(2)$ be denoted by x^i $(i=1,2,3)$ and let the dual homogeneous coordinates of a line l be denotes by l_i $(i=1,2,3)$. Let ε and η be two real numbers (*nonzero* for the moment) and, in a given coordinate system, associate with each point x the following function of ε and η:

$$\mathcal{Q}(x)=\varepsilon\eta(x^1)^2-\varepsilon(x^2)^2+(x^3)^2 \tag{1}$$

Associate with each line the function

$$\mathcal{Q}(l)=\eta l_1^2-l_2^2+\varepsilon\eta l_3^2 \tag{2}$$

(These functions are determined only up to a positive factor, because of the homogeneity of the coordinates.) The equation $\mathcal{Q}(l)=0$ is the envelope equation for the conic Q whose point equation is $\mathcal{Q}(x)=0$. These equations determine the isotropic elements for a metricization. We take the distance measure on a line to be

$$\frac{k}{\varepsilon}\ln\chi \tag{3}$$

and the angle measure about a point to be

$$\frac{k'}{\eta}\ln\chi \tag{4}$$

where χ denotes in each case the appropriate cross ratio and k and k' are constants. The denominators have been inserted in preparation for the contraction process that will give rise to parabolic measures. We find that the distance measure on a line l is elliptic, null or hyperbolic according as $\varepsilon\mathcal{Q}(l)$ is negative, zero or positive, and that the angle measure about a point x is elliptic, null or hyperbolic according to whether $\eta\mathcal{Q}(x)$ is negative, zero or positive.

In particular the set of all points satisfying $\mathcal{Q}(x)>0$ and all lines satisfying $\mathcal{Q}(l)>0$ constitute a Cayley-Klein plane in which the distance measure is elliptic or hyperbolic according as ε is negative or positive and the angle measure is elliptic or hyperbolic according as η is negative or positive.

So far, Q is nonsingular. We can, however, consider *deformations* of the conic Q obtained by varying ε and η and in particular we can consider the limit is either ε or η or both go to zero. Then $\mathcal{Q}(x)=0$ becomes the equation of a singular conic and $\mathcal{Q}(l)=0$ that of a singular conic envelope. They continue to specify isotropic elements for a metricization. The distance measure (3) on a nonisotropic line becomes a *parabolic* measure in the limit $\varepsilon \to 0$ and the angle measure (4) about a nonisotropic point becomes parabolic in the limit $\eta \to 0$. With this interpretation, we get all nine Cayley-Klein planes:

The set of all points satisfying $\mathcal{Q}(x)>0$ and all lines satisfying $\mathcal{Q}(l)>0$ constitutes a Cayley-Klein plane in which the distance measure is elliptic, parabolic or hyperbolic according as ε is negative, zero or positive and the angle measure is elliptic, parabolic or hyperbolic according as η is negative, zero or positive.

The corresponding Cayley-Klein plane will be denoted by the symbol $\{\varepsilon, \eta\}$. (The planes $\{\varepsilon, \eta\}$ and $\{\alpha\varepsilon, \beta\eta\}(\alpha>0,\ \beta>0)$, are of course isomorphic.)

3. THE LIE ALGEBRAS

The isometry group of the Cayley-Klein plane $\{\varepsilon, \eta\}$ is the group of projective collineations on $P(2)$ that preserves the conic Q and its envelope. It is easily seen that the corresponding matrix group that acts on the coordinate triples is generated by

$$G_1=\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & -\varepsilon \\ \cdot & -1 & \cdot \end{pmatrix}, \qquad G_2=\begin{pmatrix} \cdot & \cdot & -\varepsilon\eta \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}, \qquad G_3=\begin{pmatrix} \cdot & \eta & \cdot \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} \tag{5}$$

The commutators for the Lie algebra are therefore

$$\begin{aligned} [G_2, G_3] &= -\varepsilon G_1 \\ [G_3, G_1] &= G_2 \\ [G_1, G_2] &= -\eta G_3 \end{aligned} \tag{6}$$

(our G_1, G_2, and G_3 are Sanjuan's *H*, *P*, and *K*, respectively). We now have an intuitively appealing geometrical interpretation of the various contractions of (6), in terms of deformations of a conic and its envelope.

4. GENERALIZATION TO HIGHER DIMENSIONS

Let Q be a nonsingular hyperquadric in a real $P(N)$. Call the points on it, and the p-dimensional subspaces tangential to it, *isotropic*. Distance measure on all the lines can be imposed as for the case $N=2$. There are various angle measures.

Let ω be a $(p-1)$-space and Ω a $(p+2)$-space containing ($p=1,\ldots,N-1$). Then an angle measure can be associated with such an (ω,Ω) pair as follows: The angle between two nonisotropic p-spaces containing ω and contained in Ω is defined to be $\kappa \ln \chi$, where χ is the cross ratio that the two p-spaces make with the two isotropic p-spaces containing ω and contained in Ω. κ is a constant, depending only on p and on whether the resulting measure is elliptic or hyperbolic. The measure is null if either ω or Ω is isotropic.

It is convenient to introduce the "(-1)-space" $\varnothing$ (the *null space*) which is contained in every subspace of $P(N)$ and which is nonisotropic. The distance measure on a line l is then a measure associated with the (ω,Ω) pair with $\omega=\varnothing$ and $\Omega=1$.

Denote the homogeneous coordinates of a point x in $P(N)$ by x^i ($i=1,\ldots,N+1$). A $(p-1)$-space ω can be specified by a set of homogeneous coordinates $\omega^{i_1,\ldots,i_p}$ constructed by skewsymmetrization from the coordinates of p points in ω that are not all in a $(p-2)$-space:

$$\omega^{i_1,\ldots,i_p} = x_1^{[i_1} x_2^{i_2} \cdots x_p^{i_p]} \tag{7}$$

Let ε_0, $\varepsilon_1,\ldots,\varepsilon_{N-1}$ be a set of nonzero real numbers and write

$$\begin{aligned} Q_{N+1} &:= 1 \\ Q_N &:= -\varepsilon_0 \\ Q_{N-1} &:= \varepsilon_0\varepsilon_1 \\ Q_{N-p} &:= (-)^{p+1}\varepsilon_0\ldots\varepsilon_p \\ Q_1 &:= (-)^N\varepsilon_0\ldots\varepsilon_{N-1} \end{aligned} \tag{8}$$

Associate, with each point x, the following function of the $\varepsilon_0,\ldots,\varepsilon_{N+1}$:

$$\mathscr{Q}(x) := \sum_i Q_i (x^i)^2 \tag{9}$$

More generally associate with each $(p-1)$-space $\omega (p=1,\ldots,N+1)$ the function

$$\mathcal{Q}(\omega) := \frac{\sum_{i_1 \ldots i_p} Q_{i_1} \ldots Q_{i_p} (\omega^{i_1 \ldots i_p})^2}{\prod_{j=N-p+2}^{N+1} Q_j} \tag{10}$$

(These functions are defined only up to a constant positive factor, because of the homogeneity of the coordinates.) It is convenient to define also $\mathcal{Q}(\varnothing) = \mathcal{Q}(P) = 1$, where P is the whole $P(N)$.

Let Q be the hyperquadric whose point equation is $\mathcal{Q}(x) = 0$. Then if Q is nondegenerate, the equations $\mathcal{Q}(\omega) = 0$ are the equation for the subspaces tangential to Q. For $N > 2$, a nonsingular (hyper-) quadric can be degenerate, and in this case the tangent spaces are defined to be those satisfying the equations $\mathcal{Q}(\omega) = 0$. The denominator in (10) has been inserted in preparation for the contraction process.

The set of equations $\mathcal{Q} = 0$ determine all the isotropic elements for a metricization. The measure associated with an (ω, Ω) pair is defined as

$$\frac{k_p}{\varepsilon_p} \ln \chi \tag{11}$$

where χ is the appropriate cross ratio and k_p is a constant (depending only on whether the resulting measure is elliptic or hyperbolic). We find the following:

The measure is elliptic, null, or hyperbolic according as $\varepsilon_p\, \mathcal{Q}(\omega)\, \mathcal{Q}(\Omega)$ *is negative, zero, or positive* (statement A).

Moreover, it is possible to consider the limits are various of the ε_p go to zero. In this way, we arrive at 3^N metric spaces that are generalizations of the nine Cayley–Klein planes.

The set of all subspaces of the $P(N)$ that satisfy $\mathcal{Q}(\omega) > 0$ constitutes a "Cayley–Klein N-space," in which the measure associated with each value of p ($p = 0, \ldots, N-1$) is elliptic, parabolic, or hyperbolic according as ε_p is negative, zero, or positive.

We can denote the Cayley–Klein N spaces by the symbols $\{\varepsilon_0, \ldots, \varepsilon_{N-1}\}$

The Lie algebra of the isometry group of $\{\varepsilon_0, \ldots, \varepsilon_{N-1}\}$ can be obtained as follows. Let e_{ij} denote the $n \times n$ matrix with a 1 at the intersection of the ith row and jth column, and zeros in all other positions. These matrices generate $GL(n)$ and satisfy

$$[e_{ij}, e_{kl}] = \delta_{ik} e_{il} - \delta_{li} e_{kj} \tag{12}$$

Taking the ε_p to be all nonzero for the moment, it is easy to see that the equation $\mathcal{Q}(x) = \sum_i Q_i (x^i)^2$ is preserved by the linear transformation of coordinates generated by the $GL(N+1)$ matrices

$$G_{ij} = -G_{ji} = (Q_i e_{ij} - Q_j e_{ji}) / Q_{\max(i,j)} \tag{13}$$

(The denominator is necessary in order to get the correct limit as some of the ε_p go to zero.) The commutation relations for the Lie algebra of the isometry group of $\{\varepsilon_0, \ldots, \varepsilon_{N-1}\}$ are easily obtained.

We get

$$\begin{aligned}[G_{ij}, G_{kl}] = |Q_{\max(i,j)} Q_{\max(k,l)}|^{-1} (Q_j \delta_{jk} |Q_{\max(i,l)}| G_{il} \\ - Q_i \delta_{ik} |Q_{\max(j,l)}| G_{jl} + Q_i \delta_{il} |Q_{\max(j,k)}| G_{jk} \\ - Q_j \delta_{jl} |Q_{\max(i,k)}| G_{ik})\end{aligned} \tag{14}$$

5. THE DE SITTER AND MINKOWSKI SPACES

The content of the previous section can be clarified by considering particular examples. Consider the case $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (\varepsilon, 1, -1, -1)$ in $P(4)$. The isotropic points are those whose homogeneous coordinates satisfy $\mathcal{Q}(x) = 0$, where

$$\mathcal{Q}(x) = \varepsilon(x_1^2 + x_2^2 + x_3^2 - x_4^2) + x_5^2 \tag{15}$$

According to (14), the generators P_μ and $J_{\mu\nu} = (\text{sign } \varepsilon) G_{\mu\nu}$ $(\mu, \nu = 1, \ldots, 4)$ satisfy

$$\begin{aligned}[J_{\mu\nu}, J_{\rho\sigma}] &= \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} + \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\sigma} J_{\mu\rho} \\ [J_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \\ [P_\mu, P_\nu] &= -\varepsilon J_{\mu\nu}\end{aligned} \tag{16}$$

where $(\eta_{\mu\nu}) = \text{diag}(1, 1, 1, -1)$. These are the commutation relations for the Lie algebras of the de Sitter groups [$SO(4, 1)$ if ε is positive and $SO(3, 2)$ if ε is negative].

The two de Sitter spaces familiar to physicists are those consisting of all the points for which $\mathcal{Q}(x) > 0$. They both become Minkowski space in the limit $\varepsilon \to 0$. The set of all points satisfying $\mathcal{Q}(x) < 0$ constitutes a different kind of geometry which [as is seen from the form of (15)] has no $\varepsilon \to 0$ limit.

The expressions $\mathcal{Q}(\omega)$ given by (10) for lines, planes, and hypersurfaces (3-spaces) in a de Sitter space are easily written down, and those for Minkowski space obtained from them by taking the limit $\varepsilon \to 0$. It turns out that $\mathcal{Q}(\omega) > 0$ corresponds to a timelike subspace ω and $\mathcal{Q}(\omega) < 0$ corresponds to a spacelike subspace ω. [This is most easily seen by working out $\mathcal{Q}(\omega)$ for the coordinate axes, coordinate planes, and coordinate 3-spaces of the Cartesian coordinate system in Minkowski space whose axes are the edges of the reference simplex that do not lie in the 3-space at infinity $x^5=0$.] Therefore, the points of a de Sitter or Minkowski space, together with all its *timelike* lines, planes, and 3-spaces, constitute a Cayley–Klein 4-space $\{\varepsilon+--\}$. Bearing in mind that all the subspaces of a spacelike subspace are necessarily spacelike, one can employ statement A of the previous section to deduce that every spacelike 3-space is the Cayley–Klein space $\{\varepsilon--\}$. This is of course Riemann's elliptic space, Euclidean space, or Lobachevski's hyperbolic space according as ε is negative, zero, or positive.

Consider now the case $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)=(\varepsilon, \eta, -1, -1)$. The isotropic points are those whose homogeneous coordinates satisfy $\mathcal{Q}(x)=0$, where

$$\mathcal{Q}(x)=\varepsilon\eta(x_1^2+x_2^2+x_3^2)-\varepsilon x_4^2+x_5^2 \tag{17}$$

Of course, if $\eta>0$ the geometrical situation is the one we have discussed. In terms of

$$\begin{aligned} M_i &:= -\tfrac{1}{2}\epsilon_{ijk}G_{jk}\cdot(\operatorname{sign}\varepsilon\eta) \\ L_i &:= -G_{i4}\cdot(\operatorname{sign}\varepsilon\eta) \\ P_i &:= -G_{i5} \\ E &:= -G_{45} \end{aligned} \qquad (i, j, \ldots=1, 2, 3) \tag{18}$$

equation (14) can be written in the form

$$\begin{aligned} [M_i, M_j] &= \epsilon_{ijk}M_k \\ [M_i, L_j] &= \epsilon_{ijk}L_k \\ [L_i, L_j] &= \eta\epsilon_{ijk}M_k \\ [M_i, P_j] &= \epsilon_{ijk}P_k \\ [L_i, P_j] &= \eta\delta_{ij}E \\ [M_i, E] &= 0 \\ [L_i, E] &= P_i \\ [P_i, P_j] &= \varepsilon\eta\epsilon_{ijk}M_k \\ [P_i, E] &= \varepsilon L_i \end{aligned} \tag{19}$$

For $\eta>0$ we have the Lie algebras of the two kinds of de Sitter groups $(\varepsilon=\pm1)$ and the Poincaré group $(\varepsilon=0)$. In the latter case the generators **M**, **L**, **P**, and E correspond, respectively, to rotations, boosts, space translations, and time translations in Minkowski space. In the limit $\eta\to0$, Minkowski space turns into the space-time of classical kinematics and the

Poincaré group contracts to the Galilean group. Less familiar is the limit $\eta \to 0$ when $\varepsilon \neq 0$. The de Sitter space-time turns into a curious space-time appropriate to classical kinematics in a 3-space of constant curvature. Its Lie algebra is given by setting $\eta = 0$, $\varepsilon \neq 0$ in (19).

REFERENCE

Fernandez Sanjuan, M. A. (1984). Group contraction and the Nine Cayley-Klein Geometries, *International Journal of Theoretical Physics*, **23**, 1-14.